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Axisymmetric Vacuum Fields in General Projective Relativity

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A method has been derived which enables one to obtain solutions to the stationary, axially symmetric vacuum fields in general projective relativity developed by Arcidiacono from known solutions of the vacuum field in Einstein's theory. The analogue of the Kerr solution in general projective relativity has been obtained as an example. Finally, a relation between the stationary and static axially symmetric vacuum fields in general projective relativity has been derived.

1. INTRODUCTION

Recently Arcidiacono (1984, 1986, 1987) has developed a general projective relativity (GPR) based on the DeSitter Universe with the local curvature described by the generalized Einstein equations

$$R_{AB}-\tfrac{1}{2}\gamma_{AB}R=\chi T_{AB}, \qquad (A, B=0, 1, 2, 3, 4) \tag{1.1}$$

where γ_{AB} is the 5-dimensional metric and T_{AB} is the energy tensor of the material fields (Arcidiacono, 1986). From the field equations (1.1), a good number of alternative theories of gravitation and unified field theories can be obtained. In particular, the field equations for a scalar-tensor gravitational field have been obtained which have a formal similarity with the Brans-Dicke scalar-tensor theory (Brans and Dicke, 1961; Singh and Rai, 1983; Singh and Singh, 1987).

The field equations of GPR (Arcidiacono, 1986) can be written as

$$\begin{aligned}&\hat{R}_{ik}-\tfrac{1}{2}a_{ik}\hat{R}+(3n+1)\phi^{-1}(\nabla_i\nabla_k\phi-a_{ik}\Box\phi)\\&\quad-3n\phi^{-2}[(n+1)(\nabla_i\phi)(\nabla_k\phi)+na_{ik}(\nabla_l\phi)(\nabla_s\phi)a^{ls}]\\&\quad=\chi\phi^{-2}T_{ik}\end{aligned} \tag{1.2}$$

$$\hat{R}+6n[\phi^{-1}\Box\phi+(n-1)\phi^{-2}(\nabla_l\phi)(\nabla_s\phi)a^{ls}]=-2\chi\phi^{-4}T_{00} \tag{1.3}$$

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It has been shown by Arcidiacono (1984, 1986) that given a solution of the GPR field equations for $n=0$, one can obtain a solution of GPR field equations with $n \neq 0$ by the transformations

$$\gamma_{ik} = \phi^{2n} a_{ik}; \qquad \gamma_{i0} = 0 \tag{1.4}$$
$$\gamma_{00} = \phi^{2(n+1)}$$

and

$$T_{ik} \to \phi^{-2n} T_{ik}; \qquad T_{00} \to \phi^{-2n} T_{00} \tag{1.5}$$
$$\hat{T} \to \phi^{-2n} \hat{T}$$

The quantities with a caret refer to their four-dimensional components; i, k take the values 1, 2, 3, 4; a_{ik} is the metric tensor of the four-dimensional space-time. $\hat{R} = a^{ik}\hat{R}_{ik}$, where $\hat{R}_{ik}$ is the Ricci tensor constructed from the tensor a_{ik}. For the vacuum case $T_{AB} = 0$ $(A, B = 0, 1, 2, 3, 4)$ and if we consider $n = 0$, the field equations (1.2) and (1.3) reduce to

$$\hat{R}_{ik} + \frac{1}{\phi} \phi_{;ik} = 0 \tag{1.6}$$

$$\Box \phi = 0 \tag{1.7}$$

Here a semicolon denotes covariant derivative in four-dimensional space-time. The field equations (1.2), (1.3) and (1.6), (1.7) have a formal similarity with the Brans-Dicke theory (Brans and Dicke, 1961; Singh and Rai, 1983; Singh and Singh, 1987).

In an earlier paper Arcidiacono and Singh (submitted) have shown that a Birkhoff-type theorem of general relativity is true in GPR also under the assumption that the scalar field is independent of time t.

In this paper we consider the vacuum field equations of GPR in the form (1.6) and (1.7) and derive a method which enables one to obtain solutions to the stationary, axially symmetric vacuum GPR (with $n=0$) starting from vacuum solutions to the Einstein theory. Further use of the transformation (1.4) gives a family of vacuum solutions of GPR for arbitrary n. We assume that the tensor field a_{ik} and the scalar field ϕ are functions of the space coordinates x^1 and x^2 only.

The method has been applied to Kerr solutions (Kerr, 1963; Boyer and Lindquist, 1967) and the GPR analogues of Kerr solutions have been obtained. The solutions are relevant to the study of black holes in general projective relativity. Finally, a relation between stationary and static axially symmetric solutions of GPR is derived.

2. FIELD EQUATIONS

We consider a stationary, axially symmetric space-time whose metric is of the form

$$ds^2 = e^{2U}(dt+\Omega\, d\phi)^2 - e^{2K-2U}[(dx^1)^2+(dx^2)^2] - h^2 e^{-2U}(d\phi)^2 \quad (2.1)$$

where U, Ω, K, and h are functions of x^1 and x^2 only.

For the metric (2.1) the surviving equations from the GPR field equations (with $n=0$), namely (1.6) and (1.7), are

$$2(U_2^2-U_1^2)+\frac{2K_1h_1}{h}-\frac{2K_2h_2}{h}+\frac{e^{4U}}{2h^2}(\Omega_1^2-\Omega_2^2)$$
$$+\frac{1}{h}(h_{22}-h_{11})+(p_{22}-p_{11})+(p_2^2-p_1^2)$$
$$+2p_1(K_1-U_1)-2p_2(K_2-U_2)=0 \quad (2.2)$$

$$2U_1U_2-\frac{K_2h_1}{h}-\frac{K_1h_2}{h}-\frac{\Omega_1\Omega_2e^{4U}}{2h^2}+\frac{h_{12}}{h}+p_{12}$$
$$+p_1p_2-p_1(K_2-U_2)-p_2(K_1-U_1)=0 \quad (2.3)$$

$$\Omega_{11}+\Omega_{22}-\frac{1}{h}(\Omega_1h_1+\Omega_2h_2)+4(U_1\Omega_1+U_2\Omega_2)+(\Omega_1p_1+\Omega_2p_2)=0 \quad (2.4)$$

$$U_{11}+U_{22}+\frac{1}{h}(U_1h_1+U_2h_2)+\frac{1}{2h^2}(\Omega_1^2+\Omega_2^2)e^{4U}+U_1p_1+U_2p_2=0 \quad (2.5)$$

$$h_{11}+h_{22}+h_1p_1+h_2p_2=0 \quad (2.6)$$

$$p_{11}+p_{22}+p_1^2+p_2^2+\frac{1}{h}(h_1p_1+h_2p_2)=0 \quad (2.7)$$

where $e^p=\phi$ and subscripts 1 and 2 denote partial differentiation with respect to x^1 and x^2, respectively.

3. SOLUTIONS OF THE AXISYMMETRIC STATIONARY EINSTEIN VACUUM FIELDS

Let us consider the Einstein vacuum field equations corresponding to the metric

$$ds^2 = e^{2V}(dt+\Omega\, d\phi)^2 - e^{2K-2V}[(dx^1)^2+(dx^2)^2]H^2e^{-2V}(d\phi)^2 \quad (3.1)$$

where Ω and K are the same as those given in (2.1) and H and V are functions of x^1 and x^2; the set of field equations from the Einstein vacuum field equations $R_{ij}=0$ corresponding to the metric (3.1) are

$$2(V_2^2-V_1^2)+\frac{2K_1H_1}{H}-\frac{2K_2H_2}{H}+\frac{e^{4V}}{2H^2}(\Omega_1^2-\Omega_2^2)+\frac{1}{H}(H_{22}-H_{11})=0 \quad (3.2)$$

$$2V_1V_2-\frac{K_2H_1}{H}-\frac{K_1H_2}{H}-\frac{e^{4V}}{2H^2}\Omega_1\Omega_2+\frac{H_{12}}{H}=0 \quad (3.3)$$

$$\Omega_{11}+\Omega_{22}-\frac{1}{h}(\Omega_1H_1+\Omega_2H_2)+4(V_1\Omega_1+V_2\Omega_2)=0 \quad (3.4)$$

$$V_{11}+V_{22}+\frac{1}{H}(V_1H_1+V_2H_2)+\frac{e^{4U}}{2H^2}(\Omega_1^2+\Omega_2^2)=0 \quad (3.5)$$

$$H_{11}+H_{22}=0 \quad (3.6)$$

From (2.6) and (2.7), it can be found that

$$(he^p)_{11}+(he^p)_{22}=0 \quad (3.7)$$

Equations (3.6) and (3.7) suggest the relation

$$H=he^p \quad (3.8)$$

In view of (3.8) and the substitution

$$V=U+\tfrac{1}{2}p$$

the set of equations (3.2)-(3.6) reduces to

$$2(U_2^2-U_1^2)+\frac{2K_1h_1}{h}-\frac{2K_2h_2}{h}+\frac{e^{4U}}{2h^2}(\Omega_1^2-\Omega_2^2)$$
$$+\frac{1}{h}(h_{22}-h_{11})+(p_{22}-p_{11})+\tfrac{3}{2}(p_2^2-p_1^2)$$
$$+2p_1(K_1-U_1)-2p_2(K_2-U_2)-\frac{2}{h}(h_1p_1-h_2p_2)=0 \quad (3.9)$$

$$2U_1U_2-\frac{K_2h_1}{h}-\frac{K_1h_2}{h}-\frac{e^{4U}}{2h^2}\Omega_1\Omega_2+\frac{h_{12}}{h}+p_{12}+\tfrac{3}{2}p_1p_2$$
$$-(K_2-U_2)p_1-(K_1-U_1)p_2+\frac{1}{h}(h_1p_1+h_2p_2)=0 \quad (3.10)$$

$$\Omega_{11}+\Omega_{22}-\frac{1}{h}(\Omega_1h_1+\Omega_2h_2)+4(\Omega_1U_1+\Omega_2U_2)+(\Omega_1p_1+\Omega_2p_2)=0 \quad (3.11)$$

$$U_{11}+U_{22}+\frac{1}{h}(U_1h_1+U_2h_2)+\frac{e^{4U}}{2h^2}(\Omega_1^2+\Omega_2^2)$$
$$+\frac{1}{2}\left[p_{11}+p_{22}+p_1^2+p_2^2+\frac{1}{h}(h_1p_1+h_2p_2)\right]$$
$$+(U_1p_1+U_2p_2)=0 \tag{3.12}$$

$$h_{11}+h_{22}+2(h_1p_1+h_2p_2)+h(p_{11}+p_{22}+p_1^2+p_2^2)=0 \tag{3.13}$$

On comparing equations (3.9)-(3.12) with (2.4)-(2.7), we observe that they are equivalent if the following relations hold:

$$p_2^2-p_1^2=-\frac{4}{h}(h_2p_2-h_1p_1) \tag{3.14}$$

$$\tfrac{1}{2}p_1p_2+\frac{1}{h}(h_1p_1+h_2p_2)=0 \tag{3.15}$$

$$p_{11}+p_{22}+p_1^2+p_2^2+\frac{1}{h}(h_1p_1+h_2p_2)=0 \tag{3.16}$$

Equation (3.13) together with (3.16) is equivalent to the set of equations (2.6) and (2.7). Hence the set of equations (3.9)-(3.13) along with (3.16) is equivalent to the set of equations (2.2)-(2.7) provided the conditions (3.14) and (3.15) are satisfied. Therefore, in GPR, vacuum axisymmetric stationary solutions are obtainable from the Einstein vacuum axisymmetric stationary solutions when the relations (3.14) and (3.15) are satisfied.

Now equations (3.14) and (3.15) give the relation between p and h as

$$p=-4\log h$$

Thus, finally p and h are known in terms of H. Hence we have established the result.

Theorem 1. Given any Einstein vacuum axisymmetric stationary solution $(U_E, \Omega_E, K_E, H_E)$, one can generate a corresponding GPR (with $n=0$) vacuum axisymmetric stationary solution $(U_{\mathrm{GPR}}, \Omega_{\mathrm{GPR}}, K_{\mathrm{GPR}}, H_{\mathrm{GPR}}, \phi)$, where

$$\begin{aligned} U_{\mathrm{GPR}} &= U_{\mathrm{E}}-\tfrac{1}{2}\log\phi \\ \Omega_{\mathrm{GPR}} &= \Omega_{\mathrm{E}} \\ K_{\mathrm{GPR}} &= K_{\mathrm{E}} \\ H_{\mathrm{GPR}} &= H_{\mathrm{E}}^{-1/3} \\ \phi &= H_{\mathrm{E}}^{4/3} \end{aligned} \tag{3.17}$$

Hence we have the result for GPR vacuum fields with $n\neq 0$.

Theorem 2. For every vacuum solution of GPR with $n=0$ given by (3.17) we have a corresponding family of vacuum solutions of GPR with $n\neq 0$ given by

$$\gamma_{ik}=\phi^{2n}a_{ik}, \qquad \gamma_{i0}=0$$
$$\gamma_{00}=\phi^{2(n+1)}, \qquad i,k=1,2,3,4 \tag{3.18}$$

where a_{ik} is the four-dimensional metric tensor given by (2.1).

4. ANALOGUE OF KERR SOLUTION IN GENERAL PROJECTIVE RELATIVITY

The Kerr metric is given by (Kerr, 1963; Boyer and Lindquist, 1967)

$$ds^2=-(r^2+a^2\cos^2\theta)\left(\frac{dr^2}{r^2-2mr+a^2}+d\theta^2\right)-(r^2+a^2)\sin^2\theta\, d\phi^2$$
$$+dt^2-\frac{2mr}{r^2+a^2\cos^2\theta}(dt+a\sin^2\theta\, d\phi)^2 \tag{4.1}$$

If we use a coordinate transformation similar to the one used by Mishra and Pandey (1972), we can write the metric (4.1) in the form

$$ds^2=\frac{L^2+a^2\cos^2\theta-2mL}{L^2+a^2\cos^2\theta}dt^2-(L^2+a^2\cos^2\theta)[(dx^1)^2+(dx^2)^2]$$
$$-\left[(L^2+a^2)\sin^2\theta+\frac{2mLa^2\sin^4\theta}{L^2+a^2\cos^2\theta}\right]d\phi^2$$
$$-\frac{4mLa\sin^2\theta}{L^2+a^2\cos^2\theta}dt\, d\phi \tag{4.2}$$

where

$$L=e^R+m+[(m^2-a^2)/4]e^{-R}$$

and coordinates r and R are related through

$$r=R+m+\frac{m^2-a^2}{4R}$$

In view of Theorem 1, the GPR vacuum solution (for $n=0$) for the metric (4.2) is given by

$$ds^2=\frac{L^2+a^2\cos^2\theta-2mL}{L^2+a^2\cos^2\theta}(L^2+a^2-2mL)^{-2/3}\sin^{-4/3}\theta$$
$$\times\left(dt-\frac{2mLa\sin^2\theta}{L^2+a^2\cos^2\theta-2mL}\,d\phi\right)^2$$
$$-(L^2+a^2-2mL)^{2/3}\sin^{4/3}\theta\,(L^2+a^2\cos^2\theta)(d\theta^2+dR^2)$$
$$-(L^2+a^2-2mL)^{1/3}\sin^{2/3}\theta\frac{(L^2+a^2\cos^2\phi)}{L^2+a^2\cos^2\theta-2mL}\,d\phi^2 \qquad (4.3)$$

with

$$\phi=(L^2+a^2-2mL)^{2/3}\sin^{4/3}\theta \qquad (4.4)$$

Hence the stationary axially symmetric vacuum solution of GPR (with $n\neq 0$) can be directly written by applying Theorem 2 to (4.3) with ϕ given by (4.4). (We have omitted the explicit expressions for the sake of brevity.)

5. STATIONARY AXISYMMETRIC GPR SOLUTIONS FROM STATIC AXISYMMETRIC GPR FIELDS

In this section we show that in GPR (with $n=0$) the set of equations (2.2)-(2.7) can be reduced to the set of vacuum axisymmetric static field equations

We consider an auxiliary function $\bar{L}$ given by

$$e^{-2U}=\lambda e^p\cosh 2\bar{L} \qquad (5.1)$$

and define relations between Ω and $\bar{L}$ as

$$\Omega_1=-2\lambda he^p\bar{L}_2$$
$$\Omega_2=2\lambda he^p\bar{L}_1 \qquad (5.2)$$

where λ is any arbitrary constant. With the help of (5.1) and (5.2), equation (2.4) is identically satisfied and equations (2.2), (2.3), and (2.5) reduce to

$$2(\bar{L}_2^2-\bar{L}_1^2)+\frac{2X_1h_1}{h}-\frac{2X_2h_2}{h}+\frac{1}{h}(h_{22}-h_{11})+(p_{22}-p_{11})$$
$$+2(X_1-\bar{L}_1)p_1-2(X_2-\bar{L}_2)p_2+(p_2^2-p_1^2)=0 \qquad (5.3)$$

$$2\bar{L}_1\bar{L}_2-\frac{X_1h_2}{h}-\frac{X_2h_1}{h}+\frac{h_{12}}{h}+p_1p_2+p_{12}-(X_1-\bar{L}_1)p_2-(X_2-\bar{L}_2)p_1=0 \quad (5.4)$$

and

$$\bar{L}_{11}+\bar{L}_{22}+\frac{1}{h}(\bar{L}_1 h_1+\bar{L}_2 h_2)+(\bar{L}_1 p_2+\bar{L}_2 p_2)=0 \tag{5.5}$$

respectively, where

$$\begin{aligned} X_1 &= K_1+\frac{\bar{L}_1+\frac{1}{4}p_1}{p_1+h_1/h}p_1 \\ X_2 &= K_2+\frac{\bar{L}_2+\frac{1}{4}p_2}{p_2+h_2/h}p_2 \end{aligned} \tag{5.6}$$

when the scalar $\phi=e^p$ satisfies

$$\phi_2 h_1=\phi_1 h_2 \tag{5.7}$$

If (5.7) holds, the integrability conditions for equation (5.6) require that the scalar ϕ satisfies

$$\bar{L}_1\phi_2=\bar{L}_2\phi_1 \tag{5.8}$$

Until now equations (2.6) and (2.7) have been unaffected. Hence with (5.7) and (5.8) satisfied, equations (5.3)-(5.6) along with (2.6) and (2.7) now constitute the set of GPR vacuum axisymmetric static field equations corresponding to the metric

$$ds^2=e^{2\bar{L}}(dt)^2-e^{2X-2\bar{L}}[(dx^1)^2+(dx^2)^2]-h^2e^{-2\bar{L}}(d\phi)^2 \tag{5.9}$$

Hence we have the following result in GPR.

Theorem 3. Given any axisymmetric, static GPR (with $n=0$) vacuum solution ($\bar{L}$, X, h, ϕ) with the scalar ϕ satisfying (5.7) and (5.8), one can obtain the axisymmetric, stationary GPR (with $n=0$) solution (U, Ω, K, h, ϕ) with the same scalar field ϕ. The functions U, Ω, and K in GPR are determined, respectively, from equations (5.1), (5.2), and (5.6).

Now applying the conformal transformation

$$\begin{aligned} \gamma_{ik} &= \phi^{2n}a_{ik}, \qquad \gamma_{i0}=0 \\ \gamma_{00} &= \phi^{2(n+1)}, \qquad i, k=1, 2, 3, 4 \end{aligned} \tag{5.10}$$

one can obtain the corresponding solutions of GPR field equations with $n\neq 0$.

6. CONCLUSION

The immediate utility of the theorems proved in this paper is that starting from any stationary, axially symmetric solution of Einstein's vacuum equations it is possible to generate solutions to the vacuum equations of general projective relativity with $n = 0$ and then, through a conformal transformation, for general projective relativity with $n \neq 0$. Application of Theorems 1 and 2 to the Tomimatsu–Sato solutions (Tomimatsu and Sato, 1972, 1973) results in a class of solutions of general projective relativity with parameters describing mass m, rotation a, deformation δ, and scalar field ϕ. Further, it is expected that the solutions obtained in this paper may be relevant to the study of black holes in general projective relativity.

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